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Machine Calculation of Compressible Laminar Boundary Layers

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This paper describes a practical and highly accurate method that makes use of a large-scale digital computer for solving the complete equations of steady two-dimensional and axisymmetric laminar boundary-layer flow of a compressible real gas subject to equilibrium dissociation. A conventional, dimensionless stream-function form of the equation is used. In it the x derivatives, that is, streamwise derivatives, are approximated by finite-difference equivalents. The approximation converts the partial differential equation into several of ordinary type that are then solved in succession at a number of stations along the body. The method is rapid and does not tax the capacity of the computer, because the n sets of equations corresponding to the n stations on the body are solved one after another, not simultaneously. It appears that very general boundary conditions can be treated successfully, including discontinuities. No problem of numerical stability of solutions has been found. Velocity and enthalpy profiles with three- to five-decimal-place accuracy can be obtained, depending on the problem and the demands for accuracy. For a typical problem, e.g., a blunted cone, about 24 x stations are used, and with an IBM 7094 automatic calculator, this solution requires about 10 min.

Nomenclature

c_p = specific heat at constant pressure
C = $\rho\mu/\rho_e\mu_e$
f = nondimensional stream function defined by Eq. (6)
g = total enthalpy ratio = H/H_e
h = local enthalpy
H = total enthalpy = $h + u^2/2$
k = thermal conductivity
K = bound used on values of φ', Sec. 8.1
L = count of successive solutions of the energy equation in the iterative procedure of solution
M = Mach number
n = count of number of steps in the x direction
N = $(P + 1)/2 + R$
p = pressure
P = pressure gradient parameter $(x/u_e)(du_e/dx)$
Pr = Prandtl number
q = heat-transfer rate
Q = count of successive solutions of the momentum equation in the iterative procedure of solution
r = radius of a body of revolution
R = $(x/r)(dr/dx)$
u = x component of velocity, $u/u_e = f'$
v = y component of velocity
x = distance along body surface measured from stagnation point
X = axial or chordwise distance measured from stagnation point
y = distance normal to body surface
ϵ = specified accuracy in iteration process, Table 1
η = transformed y coordinate Eq. (4), η_∞ represents effective edge of boundary layer
μ = viscosity
ν = kinematic viscosity
ρ = density
φ = $f - \eta$
ψ = stream function or $g - 1$

Subscripts

e = evaluated at edge of boundary layer
n = evaluated at station n
w = evaluated at wall
∞ = evaluated at reference condition
$(\ )'$ = differentiation with respect to η

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1.0 Introduction

SINCE the beginning of 1960, the authors have been working on the problem of solving the general equations of steady axisymmetric and two-dimensional laminar boundary-layer flow. The primary purpose has been to develop a practical, general method that makes use of a large-scale computing machine. The over-all problem really consists of two problems at high speed: 1) the physical and chemical problem of determining gas properties as a function of temperature and pressure and 2) the mathematical problem of solving the nonlinear partial differential equation system governing the motion and involving the gas properties of item 1. It is this second or mathematical problem with which we deal here.

An earlier article[1] briefly described the method as applied to incompressible flows, presented results of a number of calculations, and discussed other approaches. Reference 2 described investigations of the mathematics of the solution, and Ref. 3 described in detail the method as extended to compressible flows. The present article is based to a large extent upon this last reference, and its particular purpose is to describe the essential features of the method, which is programed in FORTRAN IV.

2.0 Governing Equations

Some of the notation and basic scheme is illustrated in Fig. 1. The basic steady equations of continuity, momentum, and energy are, with the energy equation written in terms of total enthalpy $H = h + u^2/2$:

Continuity

$$\frac{\partial}{\partial x}(r\rho u) + \frac{\partial}{\partial y}(r\rho v) = 0 \tag{1}$$

Momentum

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

Energy

$$\rho\left[u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial H}{\partial y} + \mu\left(1 - \frac{1}{Pr}\right)u\frac{\partial u}{\partial y}\right] \tag{3}$$

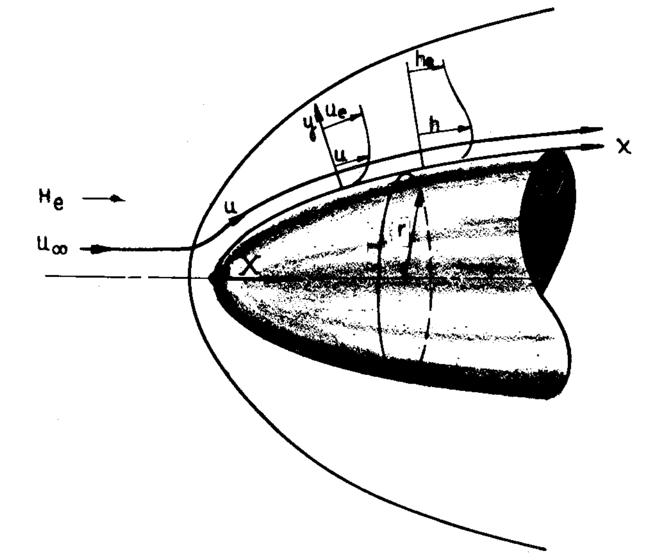

Fig. 1 Boundary layer on a body of revolution; coordinate system.

Except for minor changes in notation, these equations are identical to those given by Hayes and Probstein[4] if the Lewis-Semenov number is one. Now leave the x measure unchanged, but transform the y distance as follows, by means of a Howarth-Dorodnitsyn type of transformation:

$$\eta = \left(\frac{u_e}{\rho_\infty \mu_\infty x}\right)^{1/2} \int_0^y \rho(y)\, dy \tag{4}$$

A Mangler transformation is not used because it produces no saving in computation time or complexity. Next define a stream function such that

$$\rho r u = \frac{\partial(\psi r)}{\partial y} \qquad \rho r v = -\frac{\partial(\psi r)}{\partial x} \tag{5}$$

Finally, define a dimensionless stream function f related to ψ as follows:

$$\psi = (\rho_\infty \mu_\infty x u_e)^{1/2} f \tag{6}$$

With these relations, the momentum equation can be transformed to the following:

$$\frac{1}{C_\infty}\frac{\partial}{\partial \eta}(Cf'') + P\left(\frac{\rho_e}{\rho} - f'^2\right) + \left[\frac{P+1}{2} + R\right] ff'' - x\left[f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right] = 0 \tag{7}$$

where $C = \rho\mu/\rho_e\mu_e$, $C_\infty = \rho_\infty\mu_\infty/\rho_e\mu_e$, $P = (x/u_e)(du_e/dx)$, and $R = (x/r)(dr/dx)$. If, additionally, one introduces a quantity g defining enthalpy ratio as $g = H/H_e$, whose derivatives with respect to η are denoted by primes, the energy equation transforms to

$$\frac{1}{C_\infty}\frac{\partial}{\partial \eta}\left[\frac{C}{Pr} g' + \frac{u_e^2}{H_e} C\left(1 - \frac{1}{Pr}\right) f'f''\right] = -\left[\frac{P+1}{2} + R\right] fg' + x\left[f'\frac{\partial g}{\partial x} - g'\frac{\partial f}{\partial x}\right] \tag{8}$$

The continuity equation has been eliminated by introducing the stream function that automatically satisfies it. Equations (7) and (8) are essentially the equations that must be solved, but in the interests of error control it was found desirable to introduce translated stream and enthalpy functions as follows, where ψ now acquires a second meaning:

$$\varphi = f - \eta \qquad \psi = g - 1 \tag{9}$$

When the quantities in (9) are introduced into (7) and (8), the final equations for use in computation are obtained:

Momentum

$$\frac{\partial}{\partial \eta}(C\varphi'') = C_\infty P\left[\varphi'^2 + 2\varphi' + 1 - \frac{\rho_e}{\rho}\right] - C_\infty N(\varphi + \eta)\varphi'' + C_\infty x\left[(\varphi' + 1)\frac{\partial \varphi'}{\partial x} - \varphi''\frac{\partial \varphi}{\partial x}\right] \tag{10}$$

Energy

$$\frac{\partial}{\partial \eta}\left[\frac{C}{Pr}\psi' + \frac{u_e^2}{H_e} C\left(1 - \frac{1}{Pr}\right)(\varphi' + 1)\varphi''\right] = -C_\infty N(\varphi + \eta)\psi' + C_\infty x\left[(\varphi' + 1)\frac{\partial \psi}{\partial x} - \psi'\frac{\partial \varphi}{\partial x}\right] \tag{11}$$

where $N = R + (P + 1)/2$.

In these equations, P, N, ρ_e, u_e^2/H_e, and C_∞ are constants at any x station, but may vary from station to station along the body. The quantities C and Pr vary across the boundary layer; that is, they are functions of both x and η.

The problem is not defined until boundary conditions are specified. One condition is imposed by the requirement of no slip, i.e., $f_w' = 0$. If the wall is solid, the stream function ψ_w or f_w at the wall must be zero. But if mass transfer exists, ψ_w will not be zero, but can be arbitrary. It can be shown that the stream function and velocity v_w at the wall are related by the following expression:

$$\varphi_w = f_w = \frac{\psi_w}{(\rho_\infty\mu_\infty x u_e)^{1/2}} = -\frac{(\rho_\infty u_\infty/\mu_\infty)^{1/2}}{r(xu_e/u_\infty)^{1/2}} \int_0^x \frac{\rho_w v_w}{\rho_\infty u_\infty} r\, dx \tag{12}$$

With regard to the function g, either temperature or temperature gradient can be specified at the wall. These two conditions correspond to specifying g_w or g_w'. Outer conditions require that velocity and temperature merge smoothly into freestream values. All these boundary conditions may be summarized as follows:

At $\eta = 0$

$$\varphi_w = f_w \text{ (arbitrary)} \qquad \psi_w = g_w - 1 \text{ (arbitrary)}$$

$$\varphi_w' = -1 \qquad \text{or} \qquad \psi_w' = g_w' \text{ (arbitrary)} \tag{13}$$

At $\eta \to \infty$

$$\varphi' \to 0 \qquad \psi \to 0$$

The fundamental system to be solved is that defined by (10, 11, and 13). In (13), the word "arbitrary" signifies arbitrary values as functions of x.

3.0 Conversion to Ordinary Differential Equations

Both Eqs. (10) and (11) contain first derivatives with respect to x. These can be approximated by classical Lagrangian methods in terms of values of the function at two or more points, which need not be equally spaced. If the function of interest is called F and if it is to be evaluated at points x_n, x_{n-1}, x_{n-2}, etc., then the three-point approximation of its derivative is the formula

$$\frac{\partial F_n}{\partial x} = \frac{(2x_n - x_{n-1} - x_{n-2})F_n}{(x_n - x_{n-1})(x_n - x_{n-2})} - \frac{(x_n - x_{n-2})F_{n-1}}{(x_n - x_{n-1})(x_{n-1} - x_{n-2})} + \frac{(x_n - x_{n-1})F_{n-2}}{(x_n - x_{n-2})(x_{n-1} - x_{n-2})} + \frac{(x_n - x_{n-1})(x_n - x_{n-2})}{6}\frac{\partial^3 F(\xi)}{\partial x^3} \tag{14}$$

The last term on the right is the error term, and ξ is some value of x in the interval $x_n - x_{n-2}$. If these or higher-order expressions are used to replace $\partial\varphi'/\partial x$, $\partial\varphi/\partial x$, and $\partial\psi/\partial x$ in (10) and (11), the partial differential equations are converted to equations of ordinary type in η only, but now they involve values of φ and ψ at several x stations.

4.0 Over-All Method of Solution

The basic flow situation being considered is shown on Fig. 2. The body may be two-dimensional, or it may be a body of revolution whose radius varies in an arbitrary manner with X or x. The edge velocity ratio u_e/u_∞ may likewise have an arbitrary variation and at the wall g_w and $(\rho v)_w$ may be arbitrary as sketched. Alternatively, g_w' may be specified. Within reason, discontinuities may be allowed. Now mark off a series of x stations as indicated in the figure. Bear in mind that the equations of both two-dimensional and axially symmetric boundary-layer flow are parabolic and that downstream influences cannot be felt upstream. Now assume that solutions are known at stations $n - 1$, $n - 2$, etc., and that the problem is to find the solution at station n.

Under these conditions, in the finite-difference derivative relation such as (14), F_{n-1}, F_{n-2} are known, but F_n is unknown and remains to be found. Upon substitution of the finite-difference expression, Eqs. (10) and (11) become ordinary differential equations applying to the station x_n but containing numerical values belonging to one or more preceding stations. To illustrate, consider the simple equation of incompressible flow for the special case of equally spaced steps of length Δx, and three-point derivative formulas. Let n be suppressed, and write subscripts only for quantities belonging to the two previous stations. The equation is

$$\varphi''' - P(\varphi'^2 + 2\varphi') + N(\varphi + \eta)\varphi'' - \frac{x}{2\Delta x}\left[(\varphi' + 1)(3\varphi' - 4\varphi_{n-1}' + \varphi_{n-2}') - \varphi''(3\varphi - 4\varphi_{n-1} + \varphi_{n-2})\right] = 0 \quad (15)$$

This equation has a kind of forcing function, supplied chiefly by the solution for the $(n - 1)$th station but also, to a lesser extent, by the solution for the $(n - 2)$th station. This kind of equation, or more precisely (10) and (11), must be solved at each station, in order to satisfy the boundary conditions. The solution proceeds downstream, station by station. The method is an implicit type and, for this reason, difficulties with numerical instability do not arise. In the original machine program for simple incompressible flow, two-, three-, and four-point-derivative formulas were programed, but the four-point did not improve accuracy to a worthwhile degree. Therefore, in the program for compressible flow, only the two- and three-point formulas were used.

5.0 Method of Integration

An important advantage of the η-transformed type of equation being used is that most of the variation of boundary-layer thickness has been removed. In this measure, the thickest and thinnest technically important boundary layers seldom differ by more than a factor of 2. Then it becomes economical to integrate all x and η values with a fixed step length $\Delta\eta$.

Any of several standard methods could be used for integration, but for this work four-point extrapolation-interpolation methods recommended by Collatz[5] were used. Details are available in Ref. 3 and need not be described here, other than to mention that $C\varphi''$ in (10) and the entire first quantity in (11) were integrated as units in the first integration. The solutions were started by means of a conventional Taylor's series expansion at the wall, coupled with the use of shorter than standard η steps to maintain accuracy.

6.0 Solving the Momentum and Energy Equations When They are Coupled and Have Variable Coefficients

By the process of substituting finite-difference expressions for the several x derivatives, the partial-differential equation system is replaced by a set of n systems of ordinary differential equations corresponding to the n stations in x. But the difficulties are by no means over. At any but very low speeds, the momentum and energy equations are coupled, and the momentum equation is nonlinear. In general, the gas property parameters are variable; and if they do vary, they are functions of ψ, and then the energy equation is nonlinear too. To compound the difficulty, the problem is a two-point boundary problem for which one boundary condition exists at $\eta = \infty$.

Description of the method for satisfying boundary conditions will be postponed until the next section, but for the present assume that they can be met, and consider the process of solving the nonlinear system of equations having ψ-dependent coefficients. Solutions require finding φ, ψ, and the coefficients all at once. A method that has been used very successfully depends on the fact that (11) becomes a linear equation if gas properties can be specified in advance as functions of η. This fact suggests an iterative scheme, and the one used is really a scheme of secondary iteration within one of primary iteration. Its general nature is diagramed in Fig. 3. As can be seen from the diagram, the primary iteration cycle is used to solve the momentum equation, and secondary cycles of iteration are used to solve the energy equation. In order to get started, fluid properties are assumed, after which (10) may be solved. The quantities C_∞, P, N, ρ_e are constants, of course, for any x station, and their values are determined by the boundary conditions. The first solution of the momentum equation, together with the assumed properties, makes possible solution of the energy equation (11). In this treatment, the energy equation is linear, and because it is also of second order, two independent solutions are sufficient to form all possible solutions, a fact of great importance for meeting boundary conditions. From the solution of the energy equation, together with formulas defining fluid properties,

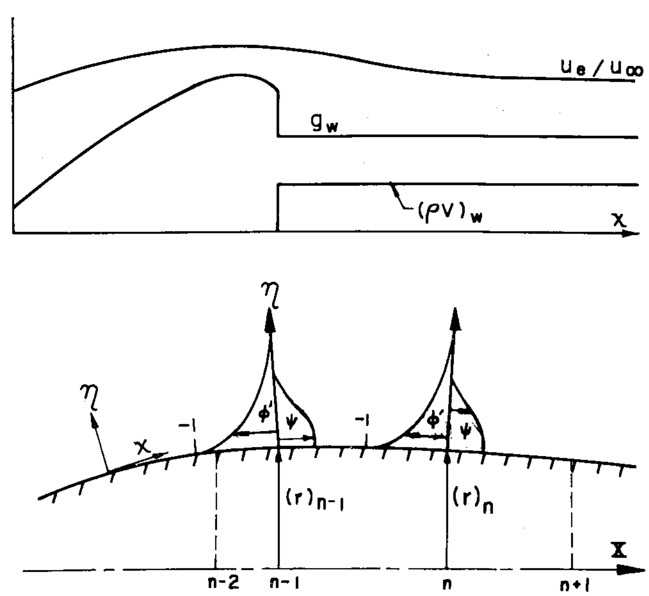

Fig. 2 Notation system for velocity and enthalpy profiles in the boundary layer on a body of revolution.

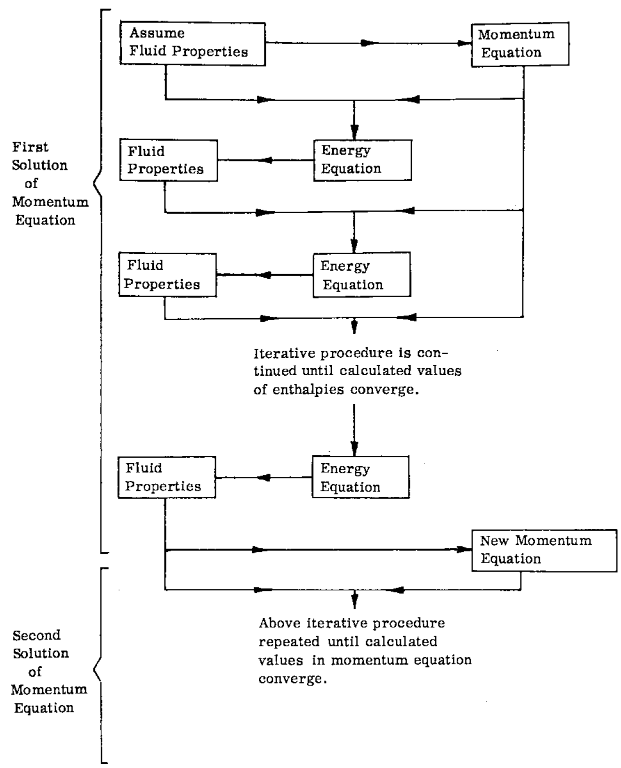

Fig. 3 Diagram of procedure for solving equations simultaneously.

new and more nearly correct fluid properties can be calculated. These are now used to obtain an improved solution of the energy equation, which, however, is still subject to φ values supplied by the first solution of the momentum equation. Again, as the diagram shows, new fluid properties are calculated, and the cycle continues until calculated values of ψ or enthalpy converge.

Secondary convergence does not represent the correct solution, but only one appropriate to the approximate solution of the momentum equation. Therefore, the improved fluid properties are used to solve anew the momentum equation, and the second cycle of the primary iteration begins. These secondary and primary cycles continue until convergence is obtained for φ. Final fluid properties found at the (n-1)th station supply very good initial values for the iteration process of the nth station.

The method converges very well. Table 1 shows the relation between convergence rate, accuracy demands, and enthalpy change ratio, either g_w or $1/g_w$, across the boundary layer. The symbol Q represents the primary iteration count, $Q = 0$ being the original calculation, $Q = 1$ the first iteration, etc.

7.0 Solution at $x = 0$

Starting the solution for $x = 0$ is very easy, and this fact is one of the important reasons for using (10) and (11) instead of the original equations (1–3). At $x = 0$, the x-factor terms are eliminated and (10) and (11) reduce to ordinary differential equations. However, the same basic procedure of solution is still required. As has already been mentioned, at stations farther downstream, the first trial values of fluid properties at a new station are just those found at the preceding station. But in the very beginning no such earlier solution exists. To get started at $x = 0$, a linear enthalpy profile that satisfies the inner and outer boundary conditions is assumed. The iteration procedure is no different from that diagramed in Fig. 3.

8.0 Meeting Boundary Conditions

Meeting boundary conditions efficiently has been the most difficult part of the entire problem. The difficulty lies primarily in the momentum equation where, since it is nonlinear, the principle of superposition of solutions cannot be used. After several attempts, a good but not excellent method has been developed.

8.1 Momentum Equation

First consider the momentum equation. In simple incompressible flow, the momentum equation (10) is the only one involved. In compressible flow, it is not the only one, but it is still isolated from the rest of the problem because of the iteration procedure. All the work of solving the compressible momentum equation, including meeting boundary conditions, takes place within the momentum equation box in the upper right-hand part of Fig. 3.

The problem is treated as an initial-value problem. There are three boundary conditions: at $\eta = 0$, $\varphi = \varphi_w$ and $\varphi_w' = -1$; and at $\eta = \infty$, $\varphi' = 0$. This last condition is replaced by φ_w''. Then a search is made for that value of φ_w'' which produces the solution having $\varphi'(\infty) = 0$. Briefly, the search is made by obtaining three solutions $\varphi(\varphi_w'')$ within bounds $\pm K$ of the correct one and then by using three-point interpolation to find the correct solution. A set of

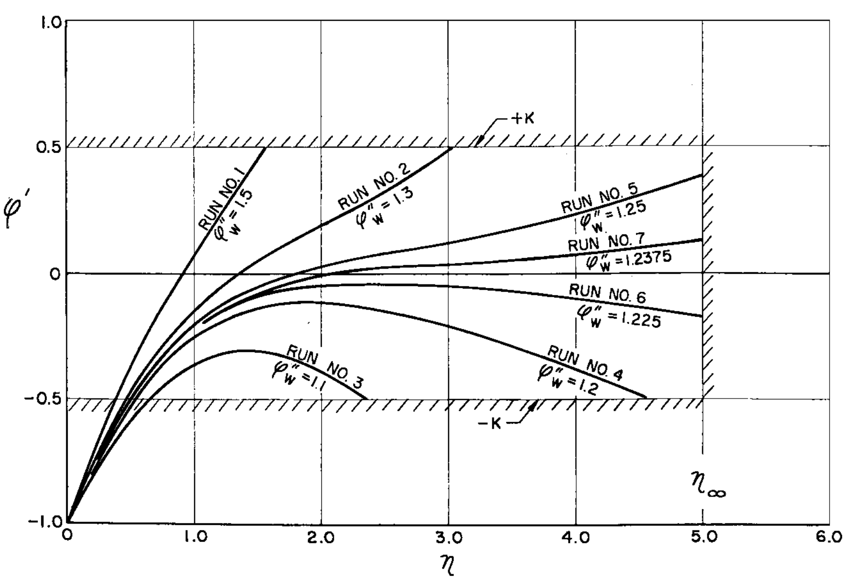

Fig. 4 Illustration of the process of searching to meet the outer boundary condition. For this example $P = 1.0$, $R = 0$, $K = 0.5$.

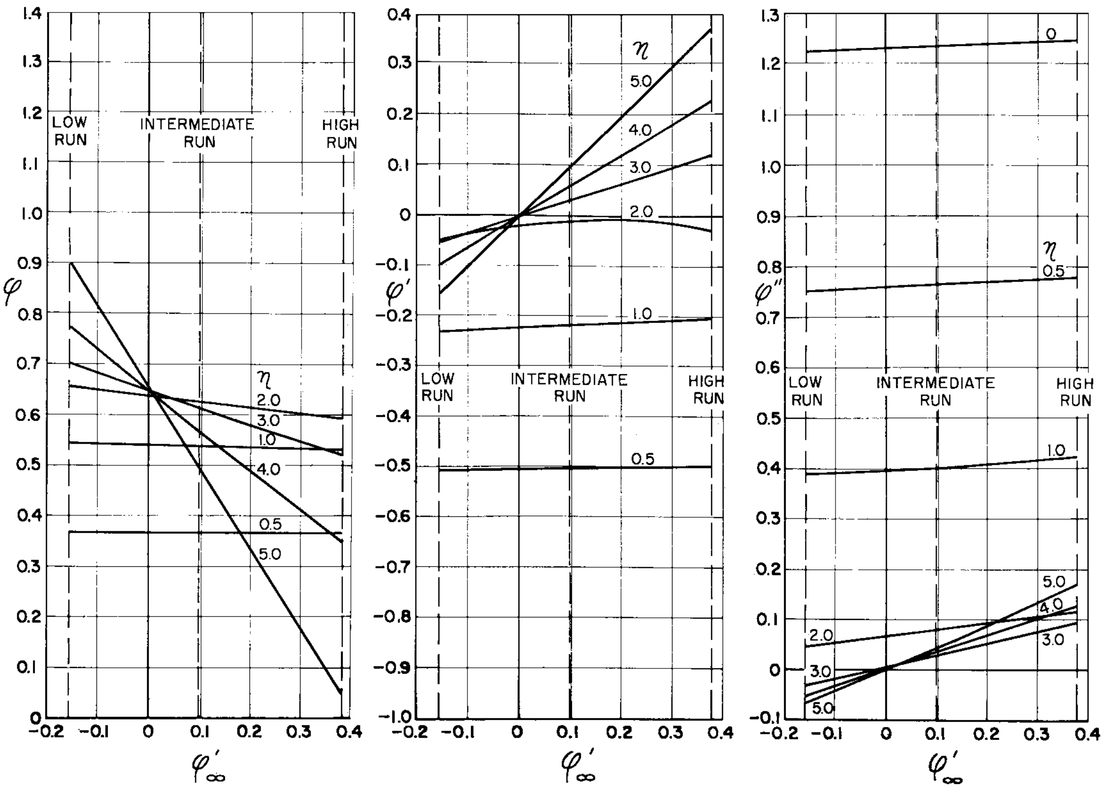

Fig. 5 Typical relations of φ, φ' φ'', with φ_∞'. For this example $P = 1.0$, $R = 0$.

trial solutions for the case of incompressible flow is shown in Fig. 4, K being 0.5.

At any fixed value of η, the computed values of φ, φ', and φ'' will be unique functions of φ_w''. Study of many cases shows that the values near the correct one deviate but little from a straight-line relation, when K is of the order of 0.5. Hence, interpolation against φ_w'' as a parameter is possible, but a method less subject to round-off difficulties was used instead. At η_∞, paired with any value of φ_w'' is a value φ_∞'. Any number that can be related to φ_w'' can just as well be related to φ_∞'. Figure 5 shows the relations for a typical case. The variations are easily seen to be describable by a parabolic interpolation function. Then, if m represents the order of the derivative and the three usable runs are 1, 2, and 3, the general interpolation function is

$$\varphi^{(m)} = \frac{(\varphi_\infty' - \varphi_{2_\infty}')(\varphi_\infty' - \varphi_{3_\infty}')}{(\varphi_{1_\infty}') - \varphi_{2\infty}')(\varphi_{1_\infty}' - \varphi_{3_\infty}')}\varphi_1^{(m)} + \frac{(\varphi_\infty' - \varphi_{1_\infty}')(\varphi_\infty' - \varphi_{3_\infty}')}{(\varphi_{2_\infty}' - \varphi_{1_\infty}')(\varphi_{2_\infty}' - \varphi_{3_\infty}')}\varphi_2^{(m)} + \frac{(\varphi_\infty' - \varphi_{1_\infty}')(\varphi_\infty' - \varphi_{2_\infty}')}{(\varphi_{3_\infty}' - \varphi_{1_\infty}')(\varphi_{3_\infty}' - \varphi_{2_\infty}')}\varphi_3^{(m)} \quad (16)$$

The correct solution is that corresponding to $\varphi_\infty' = 0$. Therefore, (16) specializes to the following form, where the subscript c denotes "correct":

$$\varphi_c^{(m)}(\eta) = k_1\varphi_1^{(m)}(\eta) + k_2\varphi_2^{(m)}(\eta) + k_3\varphi_3^{(m)}(\eta) \quad (17)$$

Table 1 Number of iterative calculations Q of momentum equation necessary to produce convergence such that $|(\varphi_w'')_Q - (\varphi_w'')_{Q-1}| < \epsilon$

Enthalpy ratio across boundary layer	$\epsilon = 10^{-6}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-3}$
Approximately 10	5	3	3	2
3 or less	3	2	1	1

where k_1, k_2, and k_3 are constants corresponding to the three fractions in (16). By means of (17), the correct solution can be found at all values of η from stored values for the three trial runs. The process just described is equivalent to reading Fig. 5 up and down the ordinate $\varphi_\infty' = 0$ for all the desired values of η.

8.2 Energy Equation

The problem of meeting boundary conditions for the energy equation is considerably easier, because it is linear as used in the iteration process. Since it is of second order, two independent solutions can supply all possible solutions. One or the other of two types of wall boundary conditions occurs, those in which ψ_w is specified or those in which ψ_w' is specified. The method for treating the first case will be described in some detail and from it the procedure for handling the problem of flow with the temperature gradient specified should be obvious.

Two separate solutions of the energy equation are computed; both begin with the correct value of ψ_w, but the two use different values of ψ_w'. Let the first solution be $\psi_1(\eta)$, with ψ_w' given by the final value at the previous station. When the solution has been calculated all the way to η_∞, examine it. If $\psi_1(\eta_\infty) > 0$, compute solution $\psi_2(\eta)$ with a lower value of ψ_w', changes of 0.1 being typical. If $\psi_1(\eta_\infty) < 0$, compute it with a higher value. In this way, the correct solution is an interpolated function that is expected to provide somewhat better accuracy than an extrapolated function.

If A and B are constants, the general solution can now be written

$$\psi = A\psi_1 + B\psi_2 \quad (18)$$

Application of boundary conditions at the wall and at η_∞ then gives

$$\begin{aligned}\psi(\eta)_{\text{correct}} &= A\psi_1 + (1 - A)\psi_2 \\ \psi'(\eta)_{\text{correct}} &= A\psi_1' + (1 - A)\psi_2'\end{aligned} \quad (19)$$

Table 2 Effect of step size $\Delta\eta$ on accuracy of solution[a]

Flow		Values of f_w''				
P	g_w	$\Delta\eta = 0.5$	$\Delta\eta = 0.2$	$\Delta\eta = 0.1$	$\Delta\eta = 0.05$	$\Delta\eta = 0.02$
1.0	1.0	1.*270*[b]	1.232*70*	1.2326*27*	1.23259*1*	1.232588
0.333333	0.2	0.*540*	0.53*440*	0.5*3690*	0.534*770*	0.5347*86*
−0.0476191	2.0	0.1*905*	0.12*34*	0.123*91*	0.1247*80*	0.1247*03*
−0.1304348	0.2	...[c]	0.1*69*	0.13*91*	0.1384*25*	0.1382*56*

[a] $\eta_\infty = 9.0$, $K = 0.5$, $L = 3$, $\epsilon = 10^{-6}$.
[b] Italicized digits may not be significant.
[c] Program unable to converge to a solution.

where

$$A = -\psi_2(\eta_\infty)/[\psi_1(\eta_\infty) - \psi_2(\eta_\infty)] \quad (20)$$

This method has generally been very satisfactory. The only time trouble has occurred is at very high Mach numbers, 20 or more, in which case ψ_1 and ψ_2 may become so large at η_∞ that considerable loss in accuracy occurs in the correct solution (19) because of roundoff. The remedy has been to rerun, with values ψ_{w_1}' and ψ_{w_2}' closer to the correct values, as indicated by the first (but not sufficiently accurate) calculation. All integration, interpolation, and extrapolation processes are programed in double precision arithmetic.

9.0 Gas Properties

To describe the method fully a few words about gas properties are desirable. There is no requirement that properties such as Pr be constant, but of course, fairly simple algebraic formulas for them will save time compared with more involved ones. Sutherland's viscosity law, power laws, and more complicated viscosity laws present no difficulty. The results to follow for compressible flows use expressions developed by Cohen,[6] of which one is given to show a typical form:

$$C_\infty = \frac{\rho_\infty \mu_\infty}{\rho_e \mu_e} = \frac{(h_e/h_{\text{ref}})^{0.3329} - 0.020856}{(h_\infty/h_{\text{ref}})^{0.3329} - 0.020856} \left(\frac{p_\infty}{p_e}\right)^{0.992}$$

Equilibrium dissociation can be treated readily by use of an effective Prandtl number.

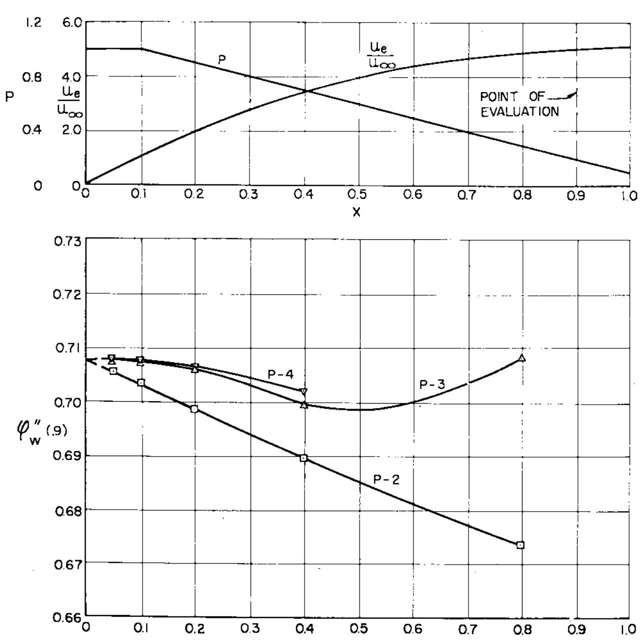

Fig. 6 Effect of step size on accuracy for a flow with decreasing P. Comparison of three methods at one x station.

10.0 Character of the Solution as Learned from Trial Runs

Accuracy and general behavior of the method of solution have been investigated by making numerous runs. A large number of known exact solutions have been recalculated as test cases. Accuracy has also been evaluated for many more problems by running convergence tests. In the case of incompressible flow, two-, three-, and four-point finite-difference derivative formulas have all been programed, and comparisons of one against another have given additional checks.

It is seen in the development of the method that several arbitrary constants exist. The principal ones are 1) step lengths normal to the flow $\Delta\eta$, 2) the "practical" edge of the boundary layer η_∞, 3) the high-low searching bound K, 4) the convergence criterion ϵ, and finally 5) the step length in the direction of the flow Δx. Table 2 shows the effect of step size $\Delta\eta$ on accuracy for a number of similar flows. The quantity f_w'', which is a measure of shear stress at the wall, has been found to be the most sensitive indicator. The flow $P = 1$ is a stagnation flow and those with $P < 0$ are decelerated flows. For this table, η_∞, K, L, and ϵ have been so chosen that the listed values truly represent variations caused solely by variation of $\Delta\eta$. The table shows that step lengths of 0.05 or 0.10 are satisfactory for most purposes.

As has already been described, the so-called correct solution is that for which φ' and ψ' equal zero at η_∞. The combination formulas (17) and (19) inherently satisfy these conditions. But if η_∞ is too low, the profiles will have finite higher derivatives, but, of course, in an acceptable solution the values of the higher derivatives must be negligible. Table 3 shows the effects of η_∞ on values of f_w'' for a stagnation flow and for a decelerating flow.‡ On accelerating flows $\eta_\infty \approx 8$ is sufficient; in fact, in Fig. 4, $\eta_\infty = 5$ was ample. On decelerating flows it should be about 12. If η_∞ is set too large, trial-solution values may become too large to handle. Values of $\varphi' = 10^{10}$ can easily be reached.

The constant K determines the base over which the three-point interpolation formula is applied. If the base is too large, the function being approximated deviates significantly from a parabola and accuracy deteriorates. Table 4 presents results of a study with the same two similar flows used previously. For the usual accuracy demands, $K = 0.5$ is seen to be about right.

The convergence criterion ϵ has already been discussed in Sec. 6.0. Therefore consider next the question of x step length. Since no concise statement about step length-accuracy relations can be made, two examples, both incom-

‡ In Tables 2, 3, and 4 certain combinations of $\Delta\eta$, η_∞, and k are common. Therefore it is to be expected that the computed results for these cases in the three tables would be identical. But inspection shows that they are not. The reason is that the data for the tables were obtained very early when the computational method was still under development and small changes were being made in details of the program. Each of the three tables then is internally consistent, but not exactly consistent with each other.

Table 3 Effect of input η_∞ on accuracy of solution[a]

	f_w''	
η_∞	$P = 1.0$, $g_w = 1.0$	$P = -0.0476191$, $g_w = 2.0$
12	1.232588	0.12477*8*[b]
10	1.232588	0.12478*0*
8	1.232588	0.1248*2*
6	1.23259*1*	0.12*69*
4	1.232*8*	0.134*2*

[a] $\Delta\eta = 0.05$, $K = 0.5$, $L = 3$, $\epsilon = 10^{-6}$.
[b] Italicized digits may not be significant.

pressible, will be presented to exhibit the relations. In neither example are there any analytic solutions, and thus, the correct answer can only be inferred from convergence runs. The first example is one having a discontinuity in slope of P, Fig. 6. It is one in which P is decreasing. Similar calculations have been made for P increasing, and for these the accuracy is better than in the example presented. Calculations have been made with steps of the length noted in Fig. 6 by the symbols for two-, three-, and four-point derivative formulas, identified in the figure as P-2, P-3, and P-4, respectively.

The results have been extrapolated to zero step length Δx, and a particular value was selected as exact. Then errors for the three formulas at $x = 0.9$ were calculated. They are plotted in Fig. 7. It shows that errors decrease approximately as Δx, $(\Delta x)^2$, and $(\Delta x)^3$, respectively, for the two-, three-, and four-point method. This variation is to be expected from analysis of errors in the three derivative formulas. In this example it is clear that 10 to 20 steps will satisfy the usual accuracy demands.

A more severe test is provided by the example in Fig. 8, which represents not a discontinuity in the slope of P but a discontinuity in P itself. Figure 9 is a crossplot of values at $x = 0.14$, the end of the calculation. Reasonable convergence at this distance appears to have been obtained by dividing the region downstream of the discontinuity into about 10 steps when the three- and four-point forms were used.

The factor Δx does not appear by itself; but always in the combination $x/\Delta x$ [see Eq. (15)]. If $x/\Delta x$ becomes too large,

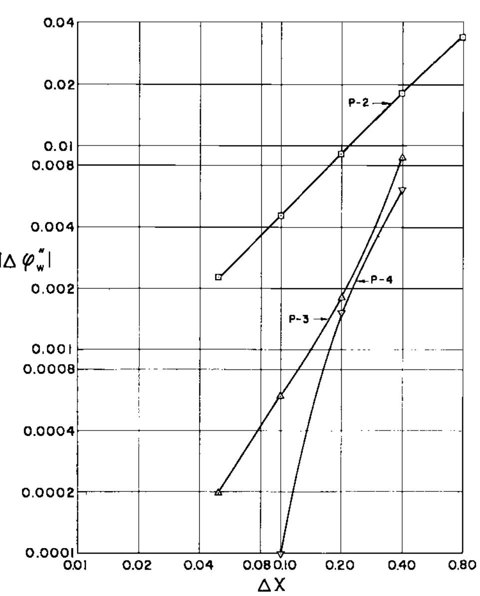

Fig. 7 A flow with decreasing P, the flow of Fig. 6. Logarithmic plot of error at one x station as a function of step length for the three methods.

Table 4 Effect of the bound K on accuracy of solution[a]

	f_w''	
K	$P = 1.0$, $g_w = 1.0$	$P = -0.0476191$, $g_w = 2.0$
0.02	1.232588	0.124780
0.2	1.232588	0.124780
0.5	1.23259*1*[b]	0.12478*6*
1.0	1.2326*0*	0.1248*1*
2.0	1.2326*2*	0.1248*9*

[a] $\Delta\eta = 0.05$, $\eta_\infty = 9.0$, $L = 3$, $\epsilon = 10^{-6}$.
[b] Italicized digits may not be significant.

the trial solutions develop such an extreme sensitivity to the initial conditions, such as φ_w'', that it may be impossible to obtain three trial solutions that stay within the $\pm K$ bounds at η_∞. One alternative is to reduce η_∞ and accept some loss in accuracy. However, the limitation is not serious, because in the large majority of problems, step lengths providing the desired accuracy can be obtained with $x/\Delta x$ well under its sensitivity bound. A safe working condition is $\Delta x \geqq 0.02\,x$. A number of successful runs have been made, however, with Δx as small as $0.01\,x$.

11.0 Solution of Three Problems

To exhibit further the accuracy and capability of the method, three problems are discussed. Two have accurate

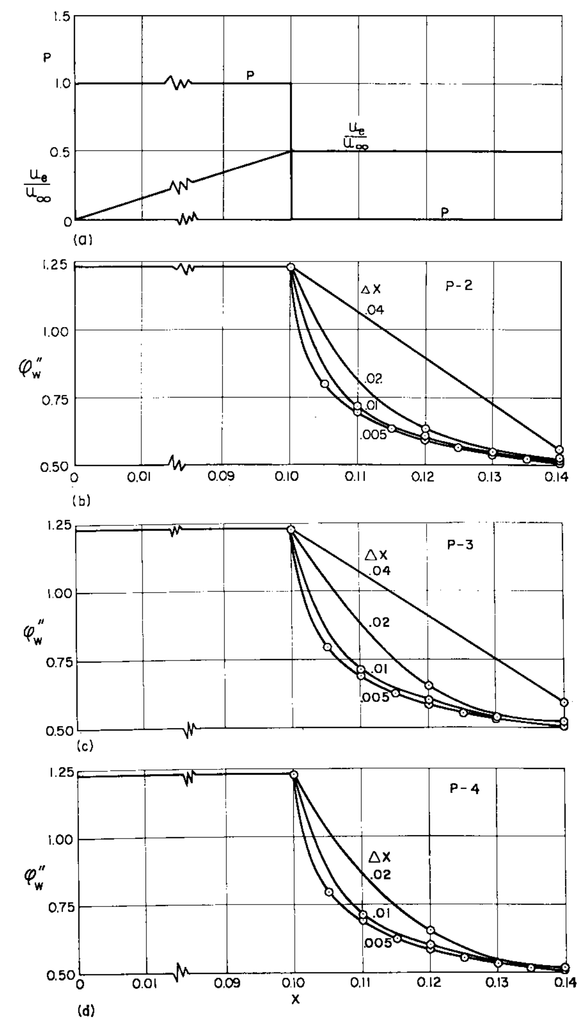

Fig. 8 A flow with discontinuity in P. Effect of step size for the three methods.

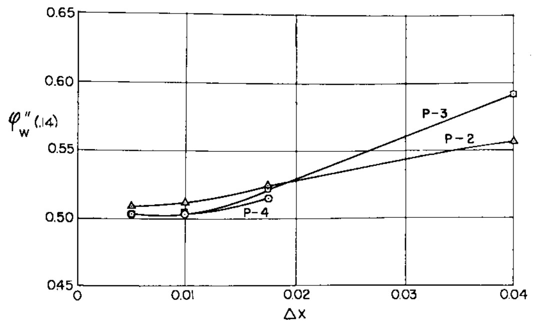

Fig. 9 A flow with discontinuity in P, the flow of Fig. 8. Effect of step size on results for the three methods at one x station.

solutions by other methods, and the third has experimental results.

11.1 Incompressible Flow on a Blunt 45° Wedge

This example is included because a Blasius type of solution is available that has moderately good convergence even at $x = \infty$. Most such series solutions are valid only near the origin. The solution is due to Van Dyke.[7] By conformal mapping he transformed inviscid, plane stagnation flow to flow about a blunted wedge of 45° half-angle. The Cartesian coordinates of the wedge in terms of a parameter ξ are

$$x = \tfrac{1}{3}a(1 + \xi^2)^{3/4} \cos(\tfrac{3}{2} \tan^{-1}\xi)$$
$$y = \tfrac{1}{3}a(1 + \xi^2)^{3/4} \sin(\tfrac{3}{2} \tan^{-1}\xi) \quad (21)$$

The term a represents the radius of the nose. Van Dyke then gives the following solution for the local skin-friction coefficient c_f, in terms of the parameter ξ:

$$c_f\left(\frac{u_\infty a}{\nu}\right)^{1/2} = 2^{3/2}\left[1.2325877\left(\frac{\xi^2}{1+\xi^2}\right)^{1/2} - 0.2469202\left(\frac{\xi^2}{1+\xi^2}\right)^{3/2} - 0.0339511\left(\frac{\xi^2}{1+\xi^2}\right)^{5/2} - 0.0108162\left(\frac{\xi^2}{1+\xi^2}\right)^{7/2} - 0.004754\left(\frac{\xi^2}{1+\xi^2}\right)^{9/2} - 0.002515\left(\frac{\xi^2}{1+\xi^2}\right)^{11/2} - \ldots\right] \quad (22)$$

It is important to note that the quantity $\xi^2/(1 + \xi^2) \rightarrow 1$ as $x \rightarrow \infty$, and so convergence seems established. More terms in the series cannot readily be written, because they depend on the Universal Series Solutions of Tifford.[8]

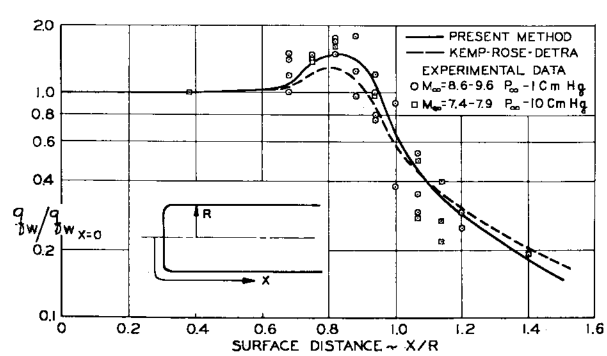

Fig. 10 Comparison of calculated and measured heat transfer on a flat-nosed cylinder in the presence of dissociation. In present method, Cohen's Pr_{eff} is used. Measurements and theory from Ref. 10 are also shown. Present calculations apply to p_∞ = 1 cm Hg.

This flow was solved by the present method to a point far back on the body, compared with the length of its nose radius, or, more precisely, to $\xi = 20$, which corresponds to $2s/a = 60.39056$, where s is surface distance measured from the stagnation point. Eighty-one steps were taken to reach this point. Table 5 presents selected values as calculated both by the general method and by (22). Supplementary studies indicate that the truncation error is about as noted in column 5. Exact values are less than those supplied by (22), which is to be expected because all terms in the series except the first one are negative. Solutions of the Falkner-Skan equation give the correct asymptotic value as $c_f(u_\infty a/\nu)^{1/2} = 2.623876$. The Douglas solution extrapolates in a $\xi/(1 + \xi^2)^{1/2}$ plane to a value of 2.62389 ± 0.00004. Van Dyke's series sums to 2.640709. Hence, this problem is apparently solved more accurately by the machine method than by the series method. Also noteworthy is the fact that 81 steps were taken in the calculation without difficulty.

11.2 Compressible Flow on a Flat Plate with Variable Temperature

Chapman and Rubesin[9] developed a method for calculating exactly the compressible laminar boundary layer on a flat plate with variable wall temperature. The method requires that the surface temperature distribution be expressible as a polynomial in surface distance, that the viscosity vary linearly with temperature through the boundary layer, and that the Prandtl number be constant. One of the examples presented by Chapman and Rubesin was calculated by the present method with the same assumptions for fluid properties. The distribution of surface temperature is given by

$$\frac{T_w}{T_e} - \frac{T_{ad}}{T_e} = \frac{T_{ad}}{T_e}(0.25 - 0.83x + 0.33x^2) \quad (23)$$

where $T_{ad}/T_e = 1 + 0.169\ M_\infty^2$ is the wall recovery temperature and T_e is the temperature at the edge of the boundary layer. Since constant c_p is assumed, the wall enthalpy distribution has the same form as the temperature. The external flow has a Mach number of 3.0. Results of the two calculations for values of $g_w'/(g_w)_{ad}$ agree within about 0.4%. This small error, which is rather consistent throughout the length of the flow, seems to come from slightly different values of the recovery factor used in the two methods.

11.3 Calculated and Measured Heat Transfer on a Blunt Body Where Dissociation Is Occurring

This third example represents a very high enthalpy flow obtained in a shock tube.[10] Experimental data were obtained at Mach numbers from 7.4 to 9.6 on a flat-nosed cylinder with a corner radius equal to one-fourth of the cylinder radius, Fig. 10. Undoubtedly, dissociation existed in the flow but whether or not it was in equilibrium is not known. Figure 10 compares the measured heat-transfer data with those calculated by the present method and the "local similarity" method developed by the authors of Ref. 10. Both methods agree with the experimental results but it appears that the present one is somewhat better.

12.0 Concluding Remarks

The method described is now in routine use and many hundreds of calculations have been made. Setting up the problem requires only the preparation of a simple table of velocities, temperatures, and body radii as a function of x. In most problems 15 to 25 tabular entries are sufficient. The method is then almost entirely automatic, and no difficulties with stability have been found. The only place where some personal assistance may be needed is in problems of extremely high-speed flow where the enthalpy ratio across the boundary layer may be 10 or more. In that case, the

Table 5 Douglas and series solutions for boundary-layer flow along a blunt 45° wedge

		$c_f(u_\infty a/\nu)^{1/2}$		
ξ	$\frac{2s}{a}$	Van Dyke [Eq. (22)]	Douglas	Approximate truncation Error [Eq. (22)]
0.02	0.020001	0.069706	0.069706	0.00000
0.10	0.100083	0.346209	0.346210	0.00000
0.20	0.200661	0.678420	0.678421	0.00000
0.70	0.725945	1.860858	1.860846	0.00004
1.60	1.838311	2.474519	2.473436	0.002
4.41	6.810822	2.625660	2.615476	0.01
9.81	21.19835	2.638022	2.622971	0.02
20.00	60.39056	2.640080	2.623691	0.02
∞	∞	2.640709	2.62389[a]	0.02

[a] Extrapolated.

method of linear combination described in Sec. 8.2 may encounter large-number difficulties and may necessitate a rerun in order to get accuracy.

In general, it appears that almost any kind of laminar, two-dimensional or axially symmetric, boundary-layer problem can be solved. Gas properties may be arbitrary and quite variable, and any kind of edge and wall boundary condition can be handled, including discontinuities. Separation can be determined very accurately. For completeness though, one type of problem must be mentioned that cannot now be solved satisfactorily. It is a problem in which rapid changes occur in a very short distance compared to the distance from the stagnation point. In such a case the factor $x/\Delta x$ [see (15)] is large, greater than 50, with the result that trial solutions grow at such a rapid exponential rate that a solution correctly meeting boundary conditions cannot be obtained. Order-of-magnitude analysis shows that the boundary-layer equations are still valid in this case; the trouble lies in the method.

An example of the kind of problem in which the method encounters trouble is that of laminar separation. If separation does exist, the basic flow undergoes very rapid changes in pressure in the last few percent of the length of the retarded flow. Then, to account for the rapid variations in pressure, steps are needed such that $x/\Delta x$ has to be of magnitude 100 to 200. At present this is impossible.

Of course, at the separation point, the boundary-layer equations break down, too. For example, in Howarth's well-known retarded flow, $U = 1 - x/8$, separation occurs at $x = 0.96$. From physical considerations it is known that the boundary-layer equations are not valid at this point. However, the method can solve Howarth's problem without difficulty to $x = 0.95$, leaving a region of only about 1% in which the method is breaking down even though the boundary-layer equations are still valid. But even here the method is able to compute skin-friction values as far as $x = 0.9589$ and, by extrapolation, separation is predicted at $x = 0.960$. However, the velocity profiles could not be found all the way to the edge of the boundary layer at the very last station calculated. Facts like these indicate that the present method is almost, but not 100%, capable of handling any problem for which the boundary-layer equations themselves are valid.

The last item mentioned and a very important one is computing time. Numerous runs show that a typical 24-station solution on an IBM 7090 computer requires about 10 min. A 24-station calculation usually gives almost four-place accuracy.

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